$$F = 5.0 - 1.9\left[\sin^2\left(\frac{\pi(X - X_m - 12.5)}{2(90 - X_m - 12.5)}\right) - \sin^2\left(\frac{12.5\,\pi}{2(90 - X_m - 12.5)}\right)\right]F_2(Y) + 0.3(Z - Z_{\min}) - 0.425 \quad \text{for } Z > Z_{\min} \tag{2a}$$

or

$$-0.17(Z - Z_{\max}) + 0.775 \text{ for } Z < Z_{\min} \tag{2b}$$

where F = radiance in the 15 μm band measured by the spacecraft Earth sensor, expressed in $\mathrm{Wm^{-2}\,sr^{-1}}$.

Figure 1 shows the radiance model for January 1, April 1, July 1, and October 1 for the year 1990. These plots are typical. Radiance values generated by the model for the year 1988 are listed in Table 1. In case of onboard correction, the data can be stored in table lookup form. Alternatively, since the radiance model is in algebraic form, which can be computed using microprocessors, it enables onboard correction of the systematic errors of the horizon sensor. This will need, additionally, the use of the onboard timer, orbit generator programs, and mathematical routines for locating the coordinates of the horizon. Deviations from the longitudinally averaged radiance caused by mesoscale weather patterns, sudden polar stratospheric warmings, etc., are not accounted for in the model.

Conclusion

This Note has proposed a radiance model for use in planning, simulating, and testing Earth sensor-based attitude determination software for postfacto orbit refinement, and also for the implementation of attitude determination and control schemes onboard spacecraft. An equation is developed for the latitudinal, daily, and yearly variation of radiance. Using this equation either in table lookup form or by numeric processing onboard, correction for the radiance gradient effect can be incorporated by suitable transformation of the radiance gradient to roll and pitch errors.

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On the Method of Matched Asymptotic Expansions

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Introduction

SINGULAR perturbation problems, where suppression of a small parameter affects the order of the problems, have been solved by a wide variety of techniques.[1-5] Two of these techniques, the singular perturbation method (SPM)[1,5] and the method of matched asymptotic expansions (MAE),[2,3] have been developed independently to a reasonable level of satisfaction. Essentially, SPM consists of expressing the total solution in terms of an *outer* solution, an *inner* solution, and an *intermediate* solution. On the other hand, in the method of MAE, a composite solution is constructed as the outer solution, the inner solution, and a *common* solution.

In this Note, a critical examination of the MAE method reveals that the various terms of the common solution of MAE can be generated as polynomials in stretched variables without actually solving for them from the outer solution as is done presently. This also shows that the common solution of the MAE method and the intermediate solution of SPM are the same and, hence, that these methods give identical results for a certain class of problems. An illustrative example is given.

Method of Matched Asymptotic Expansions

The method of matched asymptotic expansions has been used extensively in fluid mechanics.[2] In this method, a composite solution is expressed as an outer solution, plus an inner solution, and minus a common solution

We describe briefly the MAE method as applicable to initial-value problems. Consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,z,\epsilon,t) \tag{1a}$$

$$\epsilon\,\frac{\mathrm{d}x}{\mathrm{d}t} = g(x,z,\epsilon,t) \tag{1b}$$

where x and z are n- and m-dimensional state vectors, respectively, and ϵ a small positive parameter responsible for singular perturbation. We begin by representing the solutions in the form of a series in powers of ϵ as

$$x(t,\epsilon) = \sum_{i=0}^{\infty} x^{(i)}(t)\,\epsilon^i \tag{2a}$$

$$z(t,\epsilon) = \sum_{i=0}^{\infty} z^{(i)}(t)\,\epsilon^i \tag{2b}$$

and determine the various terms $x^{(i)}(t)$ and $z^{(i)}(t)$ by means of formal substitution of Eqs. (2) in Eqs. (1) and comparison of coefficients of equal powers of ϵ. Then the following set of

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recursive equations is obtained. For zeroth-order approximation,

$$\frac{dx^{(0)}(t)}{dt} = f^0\,[x^{(0)}(t),\, z^{(0)}(t),\, 0,\, t] \tag{3a}$$

$$0 = g^0\,[x^{(0)}(t),\, z^{(0)}(t),\, 0,\, t] \tag{3b}$$

and, for first-order approximation, we have

$$\frac{dx^{(1)}(t)}{dt} = f^1\,[x^{(1)}(t),\, z^{(1)}(t),\, x^{(0)}(t),\, z^{(0)}(t),\, t] \tag{4a}$$

$$\frac{dz^{(0)}(t)}{dt} = g^1\,[x^{(1)}(t),\, z^{(1)}(t),\, x^{(0)}(t),\, z^{(0)}(t),\, t] \tag{4b}$$

where notations f^0 and f^1 are used to indicate all the terms on the right-hand side. Since the series of Eqs. (2) corresponds to the solution outside the boundary layer, it is called an *outer* series.

The solution of Eqs. (3) is obtained by using $x(t=0) = x(0)$ and, in general, $z^{(0)}(t=0) \neq z(0)$. On the other hand, the solution of Eqs. (4) poses a problem, since the initial condition $x^{(1)}(t=0)$ is not yet known. Once $x^{(1)}(t)$ is solved for, $z^{(1)}(t)$ is automatically known from Eq. (4b). In order to relate the outer series of Eqs. (2) to the solution of Eqs. (1) in the boundary layer, we use a *stretching* transformation

$$\tau = t/\epsilon \tag{5}$$

Then using Eq. (5) in Eqs. (1), the stretched or *inner* problem becomes

$$\frac{d\bar{x}(\tau)}{d\tau} = f\,[x(\tau),\, z(\tau),\, \epsilon,\, \epsilon\tau] \tag{6a}$$

$$\frac{d\bar{z}(\tau)}{d\tau} = g\,[x(\tau),\, z(\tau),\, \epsilon,\, \epsilon\tau] \tag{6b}$$

This has inner series expansions of the form

$$\bar{x}(\tau,\epsilon) = \sum_{i=0}^{\infty} \bar{x}^{(i)}(\tau)\epsilon^i \tag{7a}$$

$$\bar{z}(\tau,\epsilon) = \sum_{i=0}^{\infty} \bar{z}^{(i)}(\tau)\epsilon^i \tag{7b}$$

Substitution of Eqs. (7) in Eqs. (6) and comparison of coefficients for zeroth-order approximation lead to

$$\frac{d\bar{x}^{(0)}(\tau)}{d\tau} = 0 \tag{8a}$$

$$\frac{d\bar{z}^{(0)}(\tau)}{d\tau} = \bar{g}^0\,[\bar{x}^{(0)}(\tau),\, \bar{z}^{(0)}(\tau)] \tag{8b}$$

and similarly we get equations for first-order approximation. The inner problem of Eqs. (6) has initial conditions as

$$\bar{x}^{(o)}(\tau=0) = x(t=0), \quad \bar{z}^{(o)}(t=0) = z(t=0) \tag{9a}$$

$$\bar{x}^{(i)}(\tau=0) = 0, \quad \bar{z}^{(i)}(\tau=0) = 0; \qquad i > 0 \tag{9b}$$

Still, we have not resolved the problem of determining the initial value $x^{(1)}(t=0)$ of the outer problem Eqs. (4). This is done by using a matching principle of the MAE method.[2,3] Thus, the matching principle is stated as

inner expansion of outer solution $(x^o)^i$

= outer expansion of inner solution $(x^i)^o$ (10)

To any order approximation, the composite solution x_c is given by

$$x_c = x^o + x^i - (x^o)^i = x^o + x^i - (x^i)^o \tag{11}$$

where x^o and x^i are the outer and inner solutions, respectively, to any order of approximation and $(x^o)^i = (x^i)^o$ is also called the *common* solution. Similar expressions can be given for z also.

Examination of Common Solution

In this section, we will show that the common solution defined as the inner expansion of the outer solution is simply formulated as a polynomial in the stretched variable. The steps involved in obtaining the common solution are to 1) express the outer solution in the inner variable τ, 2) expand it around $\epsilon = 0$, and 3) rearrange the resulting solution in powers of ϵ. Thus, consider the outer solution as

$$x^o(t) = x^{(0)}(t) + \epsilon x^{(1)}(t) + \cdots \tag{12}$$

We express this outer solution in the inner variable $\tau = t/\epsilon$ as

$$x^o(\epsilon\tau) = x^{(0)}(\epsilon\tau) + \epsilon x^{(1)}(\epsilon\tau) + \cdots \tag{13}$$

Expanding Eq. (13) around $\epsilon = 0$, we get

$$(x^o)^i = \left[x^{(0)}(\epsilon\tau)\big|_{\epsilon=0} + \epsilon\,\frac{\partial x^{(0)}(\epsilon\tau)}{\partial\epsilon}\Big|_{\epsilon=0} + \cdots\right] + \epsilon\left[x^{(1)}(\epsilon\tau)\big|_{\epsilon=0} + \epsilon\,\frac{\partial x^{(1)}(\epsilon\tau)}{\partial\epsilon}\Big|_{\epsilon=0} + \cdots\right] \tag{14}$$

Now evaluation of function $x(\epsilon\tau)$ at $\epsilon = 0$ in the τ plane is the same as its evaluation at $t = 0$ in the t plane, and the partial derivative of function $x^{(i)}(\epsilon\tau)$, with respect to ϵ in the τ plane, is the same as its partial derivative with respect to t multiplied by τ in the t plane. Thus,

$$(x^o)^i = \left[x^{(0)}(t=0) + \epsilon\tau\,\frac{\partial x^{(0)}(t)}{\partial t}\Big|_{t=0} + \cdots\right] + \epsilon\left[x^{(1)}(t=0) + \epsilon\tau\,\frac{\partial x^{(1)}(t)}{\partial t}\Big|_{t=0} + \cdots\right]$$

$$= x^{(0)}(0) + \epsilon\left[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(0)\right] + \cdots$$

$$(x^o)^i = \bar{x}^{(0)}(\tau) + \epsilon\bar{x}^{(1)}(\tau) + \cdots \tag{15}$$

where $\bar{x}^{(0)}(\tau) = x^{(0)}(0)$, $\bar{x}^{(1)}(\tau) = x^{(1)}(0) + \tau\dot{x}^{(0)}(0)$, and the dot over x denotes differentiation of x with respect to t. Similar expressions can be obtained for the function z. Note that the intermediate solution of SPM is obtained by 1) expanding the outer solution around $t = 0$, 2) expressing it in the inner variable τ, and 3) rearranging the resulting solution in powers of ϵ.[1,5] Then, the common solution of Eqs. (16) of the MAE method is found to be the same as the intermediate solution of SPM. Thus, the outer and inner solutions being the same in the SPM and the MAE method, we clearly can see that these two methods give identical results. Essentially, this equivalence means that the expansion of the outer solution around $t = 0$ and the transformation into the τ plane is the same as the transformation of the outer solution into the τ plane first and then the expansion around $\epsilon = 0$. The main advantage of the present formulation of the common solution is that its various terms can be *generated* very easily as polynomials in τ and, hence, one need not have an explicit outer solution to arrive at the common solution.

In this way, we suggest an improved method of MAE, where the outer and inner solutions are obtained as before and the common solution is generated simply as a polynomial in the stretched variable τ, instead of evaluating it from the explicit outer solution as is done generally.[2]

Example

Consider a simple second-order system so that we can get explicit expressions for the solutions.

$$\frac{dx}{dt} = z, \qquad x(t=0) = a \tag{16a}$$

$$\epsilon \frac{dz}{dt} = -x - z, \quad x(t=0) = b \tag{16b}$$

Applying the MAE method described earlier, we summarize the results as follows. The outer solutions corresponding to Eqs. (3) and (4) are

$$x^{(0)}(t) = ae^{-t}, \quad z^{(0)}(t) = -ae^{-t} \tag{17a}$$

$$x^{(1)}(t) = [x^{(1)}(0) - at]e^{-t} \tag{17b}$$

$$z^{(1)}(t) = [-x^{(1)}(0) + at - a]e^{-t} \tag{17c}$$

The inner solutions corresponding to Eqs. (8) and (9) are

$$\bar{x}^{(0)}(\tau) = a, \quad \bar{z}^{(0)}(\tau) = -a + (a+b)e^{-\tau} \tag{18a}$$

$$\bar{x}^{(1)}(\tau) = (a+b) - a\tau - (a+b)e^{-\tau} \tag{18b}$$

$$\bar{z}^{(1)}(\tau) = -(2a+b) + a\tau + [2a + b + (a+b)\tau]e^{-\tau} \tag{18c}$$

Considering the two-term expansions only, the common solution (CS) for x is obtained as

$$(\mathrm{CS})_x = (x^i)^o = (x^o)^i \tag{19}$$

From Eq. (18), we obtain $(x^i)^o$, the outer expansion of the inner solution by first expressing the inner solution in the outer variable $t = \epsilon\tau$ and then expanding it around $\epsilon = 0$. Thus,

$$(x^i)^o = a(1-t) + \epsilon(a+b) \tag{20}$$

Next, from Eq. (17), we obtain $(x^o)^i$, the inner expansion of the outer solution, as

$$(x^o)^i = a(1-t) + \epsilon x^{(1)}(0) \tag{21}$$

Alternatively, in the present approach, we formulate $(x^o)^i$ as

$$(x^o)^i = x^{(0)}(t=0) + \epsilon[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(t=0)]$$

$$= a(1-t) + \epsilon x^{(1)}(0) \tag{22}$$

Equating Eqs. (20) and (21), we get the value of the undetermined coefficient $x^{(1)}(0) = (a+b)$. Similarly, for z, we have

$$(\mathrm{CS})_z = (z^i)^o = (z^i)^o \tag{23}$$

From Eq. (18), we obtain $(z^i)^o$, the outer expansion of the inner solution, as

$$(z^i)^o = -a(1-t) + \epsilon[-(2a+b)] \tag{24}$$

Next, we obtain $(z^o)^i$, the inner expansion of the outer solution, as

$$(z^o)^i = -a(1-t) + \epsilon z^{(1)}(0) \tag{25}$$

Alternatively, in the improved method, we formulate $(z^o)^i$ as

$$(z^o)^i = z^{(0)}(t=0) + \epsilon\,[z^{(1)}(t=0) + \tau\dot{z}^{(0)}(t=0)]$$

$$= -a(1-t) + \epsilon z^{(1)}(0) \tag{26}$$

Using Eqs. (23–25), we get the value of the undetermined coefficient, $z^{(1)}(0)$, as

$$z^{(1)}(0) = -(2a+b) \tag{27}$$

The composite solution corresponding to Eq. (12) is

$$x_c(t,\epsilon) = ae^{-t} + \epsilon\,[(a+b)(e^{-t} - e^{-t/\epsilon}) - ate^{-t}] \tag{28a}$$

$$z_c(t,\epsilon) = -ae^{-t} + (a+b)(1+t)e^{-t/\epsilon}$$

$$+ \epsilon\,[(2a+b)(e^{-t/\epsilon} - e^{-t}) + ate^{-t}] \tag{28b}$$

Conclusion

In this Note, a critical examination of the method of matched asymptotic expansion has revealed that the terms of the common solution could be generated as polynomials in a stretched variable without actually solving for them, as is done presently. We have also seen that the common solution of the method of matched asymptotic expansion is the same as the intermediate solution of the singular perturbation method and, hence, these two methods give identical results. Two examples have been given for illustration.

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Simple Scheme for the Integration of Stiff Differential Equations

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I. Introduction

STIFF differential equations arise out of modeling systems with widely differing time constants. In the case of nonlinear equations, the Jacobian has eigenvalues that are several orders of magnitude apart. In the case of an advanced

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